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Topological odd-parity superconductors

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In this letter, we investigate topological phases of full-gapped odd-parity superconductors, which are distinguished by the bulk topological invariants and the topologically protected gapless boundary states. Using the particle-hole symmetry, we introduce Z_2 invariants characterizing topological odd-parity superconductors without or with time-reversal invariance. For odd-parity superconductors, a combination of the inversion and the $U(1)$ gauge symmetry is manifestly preserved, and the combined symmetry enables us to evaluate the Z_2 invariants from the knowledge of the Fermi surface structure. Relating the Z_2 invariants to other topological invariants, we establish characterization of topological odd-parity superconductors in terms of the Fermi surface topology. Simple criteria for topological odd-parity superconductors in various dimensions are provided. Implications of our formulas for nodal odd-parity superconductors are also discussed.

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Recently, there has been considerable interest in topological phases which are characterized by the bulk topological invariants and the topologically protected gapless boundary states. The prototype of the topological phase is the integer quantum Hall states, where the band TKNN integers or Chern numbers give the integer quantum Hall effects [1], and they ensure the stability of the gapless edge states at the same time [2]. While the time-reversal symmetry breaking (TRSB) is necessary to have non-trivial Chern numbers, there exist another topological invariants called the Z_2 invariants which classify the topological phases of the time-reversal invariant (TRI) insulators [3, 4, 5, 6]. When the Z_2 invariants are non-trivial, there exist an odd number of the Kramers pairs of gapless edge modes in two dimensions, and an odd number of the Kramers degenerate band crossings (Dirac cones) on the surface in three dimensions, respectively.

The concept of topological phases is also applicable to superconducting states [7, 8, 9, 10, 11, 12] because there is a direct analogy between superconductors and insulators: The Bogoliubov de-Gennes (BdG) Hamiltonian for a quasiparticle of a superconductor is analogous to the Hamiltonian of a band insulator, and the superconducting gap corresponds to the gap of the band insulator. Indeed, the TRSB chiral p-wave superconductors have non-trivial Chern numbers, and they support topologically protected chiral gapless edge states in analogy with the integer quantum Hall states [7]. Topological phases of noncentrosymmetric superconductors and s-wave superfluids, which support non-abelian anyons, were also investigated in [8, 11, 12].

In addition to the analogous properties, there are topological features inherent to superconductors. Superconductors possess the particle-hole symmetry (PHS) exchanging the quasiparticle with the anti-quasiparticle, which provides additional topological characteristics to superconductors. In particular, for general superconductors without spin rotation symmetry, there arise extra Z_2 invariants in one dimensional TRSB and TRI systems, and an integer winding number in three dimensional TRI one [10]. As a result, the topological superconductors are characterized by the one dimensional Z_2 invariants and the two dimensional Chern number for the TRSB case, and the one and two dimensional Z_2 invariants and the three dimensional winding number for the TRI one, respectively.

In this letter, assuming the inversion symmetry in the normal state, we present a theory of topological odd-parity superconductors. For TRI single-band odd-parity superconductors, it has been revealed that the topological properties are characterized by the Fermi surface topology in the normal state [13]. Here we extend these results to general odd-parity superconductors without or with time-reversal invariance, by using the one-dimensional Z_2 invariants obtained from the PHS. We develop a method to link the Z_2 invariants to the topology of the Fermi surface, where a combination of the inversion and the $U(1)$ gauge symmetry plays an essential role. Moreover, making connections between the Z_2 invariants and the other topological invariants mentioned above, we provide characterization of topological odd-parity superconductors in terms of the topology of the Fermi surface.

In the following, we consider a general Hamiltonian H [18] for full gapped odd-parity superconducting states,

$$H = \frac{1}{2}\sum_{\boldsymbol{k}\alpha\alpha'}(c^\dagger_{\boldsymbol{k}\alpha}, c_{-\boldsymbol{k}\alpha})H(\boldsymbol{k})\begin{pmatrix} c_{\boldsymbol{k}\alpha'} \\ c^\dagger_{-\boldsymbol{k}\alpha'}\end{pmatrix},$$
$$H(\boldsymbol{k}) = \begin{pmatrix} \mathcal{E}(\boldsymbol{k})_{\alpha\alpha'} & \Delta(\boldsymbol{k})_{\alpha\alpha'} \\ \Delta^\dagger(\boldsymbol{k})_{\alpha\alpha'} & -\mathcal{E}^T(-\boldsymbol{k})_{\alpha\alpha'}\end{pmatrix}, \tag{1}$$

where $c^\dagger_{\boldsymbol{k}\alpha}$ ($c_{\boldsymbol{k}\alpha}$) denotes the creation (annihilation) operator of electron with momentum $\boldsymbol{k}$. The suffix α labels other degrees of freedom for electron such as spin, orbital

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degrees of freedom, sub-lattice indeces, and so on. $\mathcal{E}(\boldsymbol{k})$ is an hermitian matrix describing the normal dispersion of the electron. Here we assume that the system in the normal state is symmetric under the inversion $c_{\boldsymbol{k}\alpha} \to \sum_{\alpha'} P_{\alpha\alpha'} c_{-\boldsymbol{k}\alpha'}$ with $P^2 = 1$, so $P^\dagger \mathcal{E}(\boldsymbol{k}) P = \mathcal{E}(-\boldsymbol{k})$. For an odd-parity superconductor, the gap function $\Delta(\boldsymbol{k})$ satisfies $P^\dagger \Delta(\boldsymbol{k}) P^* = -\Delta(-\boldsymbol{k})$. In addition, the Fermi statistics of electron implies $\Delta^T(\boldsymbol{k}) = -\Delta(-\boldsymbol{k})$.

An important ingredient of our theory is the PHS of the BdG Hamiltonian (1),

$$CH(\boldsymbol{k})C^\dagger = -H^*(-\boldsymbol{k}), \quad C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2}$$

From (2), we can say that if $|u_n(\boldsymbol{k})\rangle$ is a quasiparticle state with positive energy $E_n(\boldsymbol{k}) > 0$ satisfying $H(\boldsymbol{k})|u_n(\boldsymbol{k})\rangle = E_n(\boldsymbol{k})|u_n(\boldsymbol{k})\rangle$, then $C|u_n^*(-\boldsymbol{k})\rangle$ is a quasiparticle state with negative energy $-E_n(-\boldsymbol{k}) < 0$. In the following, we use a positive (negative) n for $|u_n(\boldsymbol{k})\rangle$ to represent a positive (negative) energy quasiparticle state, and set

$$|u_{-n}(\boldsymbol{k})\rangle = C|u_n^*(-\boldsymbol{k})\rangle. \tag{3}$$

To define the topological invariants, we introduce the gauge fields $A_i^{(\pm)}(\boldsymbol{k}) = i\sum_{n \gtrless 0} \langle u_n(\boldsymbol{k})|\partial_{k_i}|u_n(\boldsymbol{k})\rangle$. We also denote their sum as $A_i(\boldsymbol{k}) = A_i^{(+)}(\boldsymbol{k}) + A_i^{(-)}(\boldsymbol{k})$. From (2), we have

$$A_i^{(+)}(\boldsymbol{k}) = A_i^{(-)}(-\boldsymbol{k}). \tag{4}$$

Using this and the fact that $A_i(\boldsymbol{k})$ is the total derivative of a function, we can prove that the Wilson loop of $A_i^{(-)}(\boldsymbol{k})$ along the TRI closed path C in the Brillouin zone (BZ)

$$W[\mathrm{C}] = \frac{1}{2\pi}\oint_{\mathrm{C}} dk_i A_i^{(-)}(\boldsymbol{k}) \tag{5}$$

is quantized as $e^{2\pi i W[\mathrm{C}]} = \pm 1$ [14]. Therefore, we can introduce a Z_2 invariant $(-1)^{\nu[\mathrm{C}]}$ by $(-1)^{\nu[\mathrm{C}]} \equiv e^{2\pi i W[\mathrm{C}]}$. As we will discuss later, $(-1)^{\nu[\mathrm{C}]} = -1$ $(+1)$ corresponds to a topological non-trivial (trivial) phase of the superconducting state.

Now consider the TRI closed path C_{ij} passing through the TRI momenta Γ_i and Γ_j in Fig.1. The TRI momentum satisfies $\Gamma_i = -\Gamma_i + \boldsymbol{G}$ with a reciprocal lattice vector $\boldsymbol{G}$, and because of the periodicity of the BZ, C_{ij} forms a closed path. In the following, we evaluate the Z_2 invariant $(-1)^{\nu[\mathrm{C}_{ij}]}$ along C_{ij} by developing the method in [13].

For an odd-parity superconductor, the combination of the inversion symmetry and the $U(1)$ gauge symmetry, $c_{\boldsymbol{k}\alpha} \to iP_{\alpha\alpha'} c_{\boldsymbol{k}\alpha'}$, is manifestly preserved, although each symmetry is spontaneously broken by the condensation $\Delta(\boldsymbol{k})$. Therefore, the BdG Hamiltonian $H(\boldsymbol{k})$ has the following symmetry

$$\Pi^\dagger H(\boldsymbol{k}) \Pi = H(-\boldsymbol{k}), \quad \Pi = \begin{pmatrix} P & 0 \\ 0 & -P^* \end{pmatrix}. \tag{6}$$

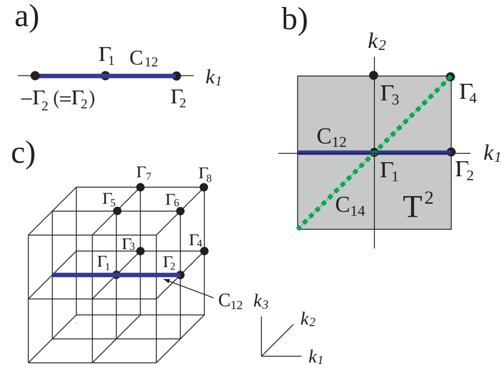

FIG. 1: The TRI momenta Γ_i, and the TRI closed path C_{ij} connecting Γ_i and Γ_j in the BZ. a) 1 dim BZ. The solid line denotes C_{12}. b) 2 dim BZ T^2. The solid line denotes C_{12}, and the dotted one C_{14}. c) 3 dim BZ. The solid line is C_{12}.

From this symmetry, we have $[H(\Gamma_i), \Pi] = 0$ for the TRI momentum Γ_i. Thus, the quasiparticle state $|u_n(\Gamma_i)\rangle$ at Γ_i is simultaneously an eigenstate of Π, $\Pi|u_n(\Gamma_i)\rangle = \pi_n(\Gamma_i)|u_n(\Gamma_i)\rangle$. Evaluation of $(-1)^{\nu[\mathrm{C}_{ij}]}$ is done by using the unitary matrices, $V_{mn}(\boldsymbol{k}) = \langle u_m(\boldsymbol{k})|\Pi C|u_n^*(\boldsymbol{k})\rangle$, and $W_{mn}(\boldsymbol{k}) = \langle u_m(-\boldsymbol{k})|C|u_n^*(\boldsymbol{k})\rangle$. Since we have $\mathrm{tr}(V^\dagger \partial_{k_i} V) = 2iA_i(\boldsymbol{k})$ from (3), $\nu[\mathrm{C}_{ij}]$ is rewritten as

$$\nu[\mathrm{C}_{ij}] = \frac{1}{\pi}\int_{\Gamma_i}^{\Gamma_j} dk_i A_i(\boldsymbol{k}) = \frac{1}{\pi i}\ln\left(\frac{\sqrt{\mathrm{det}V(\Gamma_i)}}{\sqrt{\mathrm{det}V(\Gamma_j)}}\right). \tag{7}$$

Here we have used (4) and $\mathrm{tr}(V^\dagger \partial_{k_i} V) = \partial_{k_i} \ln \mathrm{det}V$. Furthermore, because $V_{mn}(\Gamma_i)$ is recast into $V_{mn}(\Gamma_i) = \langle u_m(\Gamma_i)|\Pi C|u_n^*(\Gamma_i)\rangle = \pi_m(\Gamma_i)\langle u_m(\Gamma_i)|C|u_n^*(\Gamma_i)\rangle = \pi_m(\Gamma_i) W_{mn}(\Gamma_i)$, we obtain

$$\mathrm{det}V(\Gamma_i) = \prod_n \pi_n(\Gamma_i)\mathrm{det}W, \tag{8}$$

where $\mathrm{det}W$ is independent of Γ_i because $\partial_{k_i} \ln \mathrm{det}W = \mathrm{tr}(W^\dagger \partial_{k_i} W) = i[A_i(\boldsymbol{k}) - A_i(-\boldsymbol{k})] = 0$. Due to the PHS, $|u_n(\Gamma_i)\rangle$ and $|u_{-n}(\Gamma_i)\rangle$ share the same eigenvalue of Π and each eigenvalue appears twice in the product in (8). Therefore, taking the square root, we find $\sqrt{\mathrm{det}V(\Gamma_i)} = \prod_{n<0} \pi_n(\Gamma_i)\sqrt{\mathrm{det}W}$. As a result, (7) reduces to

$$(-1)^{\nu[\mathrm{C}_{ij}]} = \prod_{n<0} \pi_n(\Gamma_i)\pi_n(\Gamma_j), \tag{9}$$

where we have used $\pi_n^2(\Gamma_j) = 1$.

In order to attribute the Fermi surface properties to the Z_2 invariants, we make the weak-paring assumption [15]. For ordinary superconductors, the superconducting gap is much smaller than the Fermi energy. Therefore, we reasonably assume that the typical energy scale of the gap function $\Delta(\Gamma_i)$ at the TRI momentum is much smaller than that of $\mathcal{E}(\Gamma_i)$. Under this assumption, we can take $\Delta(\Gamma_i) \to 0$ without the gap closing. Because of

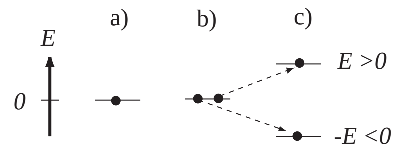

FIG. 2: Z_2 classification of edge state. a) Topologically protected zero mode. b) and c) Topologically trivial edge modes.

the topological nature of $(-1)^{\nu[\mathrm{C}_{ij}]}$, this adiabatic process does not change the value of $(-1)^{\nu[\mathrm{C}_{ij}]}$.

In the process $\Delta(\Gamma_i) \to 0$, the BdG Hamiltonian at Γ_i reduces to $H(\Gamma_i) \to \mathrm{diag}(\mathcal{E}(\Gamma_i), -\mathcal{E}^T(\Gamma_i))$. By using an eigenstate $|\varphi(\Gamma_i)\rangle$ of $\mathcal{E}(\Gamma_i)$ satisfying $\mathcal{E}(\Gamma_i)|\varphi_\alpha(\Gamma_i)\rangle = \varepsilon_\alpha(\Gamma_i)|\varphi_\alpha(\Gamma_i)\rangle$, an occupied state of $H(\Gamma_i)$ is given by $(|\varphi_\alpha(\Gamma_i)\rangle, 0)^t$ for $\varepsilon_\alpha(\Gamma_i) < 0$, and $(0, |\varphi_\alpha^*(\Gamma_i)\rangle)^t$ for $\varepsilon_\alpha(\Gamma_i) > 0$. Therefore, denoting the parity of $|\varphi_\alpha(\Gamma_i)\rangle$ as $P|\varphi_\alpha(\Gamma_i)\rangle = \xi_\alpha(\Gamma_i)|\varphi_\alpha(\Gamma_i)\rangle$, we find

$$\prod_{n<0} \pi_n(\Gamma_i) = \prod_\alpha \xi_\alpha(\Gamma_i) \prod_\alpha \mathrm{sgn}\varepsilon_\alpha(\Gamma_i), \tag{10}$$

where the sum of α is taken for all eigenstates of $\mathcal{E}(\Gamma_i)$. We notice here that the product of the parity, $\prod_\alpha \xi_\alpha(\Gamma_i)$, is independent of Γ_i since it is determined solely from $\det P$ and the dimensionality of the matrix $\mathcal{E}(\boldsymbol{k})$. Thus if we substitute (10) into (9), the contributions from the parity at Γ_i and Γ_j cancel each other, then we obtain the final expression,

$$(-1)^{\nu[\mathrm{C}_{ij}]} = \prod_\alpha \mathrm{sgn}\varepsilon_\alpha(\Gamma_i)\mathrm{sgn}\varepsilon_\alpha(\Gamma_j). \tag{11}$$

This formula is very powerful: It enables us to calculate the Z_2 invariants only from the knowledge of the band energy $\varepsilon_\alpha(\Gamma_i)$ of electron in the normal state. We also notice that the right hand side of (11) has its own topological meaning. By denoting the number of intersection points between the Fermi surface and C_{ij} as $i_0(S_\mathrm{F} \cap \mathrm{C}_{ij})$, the right hand side is found to be $(-1)^{i_0(S_\mathrm{F}\cap\mathrm{C}_{ij})/2}$ [19]. Therefore, when $i_0(S_\mathrm{F} \cap \mathrm{C}_{ij})$ is odd (even), the Z_2 invariant in (11) is non-trivial (trivial).

To see the physical meaning of the Z_2 invariants, consider a full gapped one-dimensional odd-parity superconductor. In one dimension, we have a single Z_2 invariant $(-1)^{\nu[\mathrm{C}_{12}]}$ where C_{12} is the TRI closed path in Fig.1a). When $(-1)^{\nu[\mathrm{C}_{12}]} = -1$ $(+1)$, the system is topologically non-trivial (trivial), and there exist an odd (even) number of the zero energy states on the boundary [14]. Without loss of generality, we can consider the simplest non-trivial topological phase with a single boundary zero mode. See Fig.2a). The PHS ensures the topological stability of the zero mode against small perturbation: From the PHS, a non-zero mode with energy E must be paired with a non-zero mode having the opposite energy $-E$. Therefore, a zero mode can be a non-zero mode only in pairs. This implies that the single zero mode in Fig.2a) can not aquire non-zero energy and it is stable against small perturbation. The Z_2 nature of the topological phase is evident if we add another zero mode as depicted in Fig.1b). In this case, we have a pair of zero modes, so they can be non-zero modes by small perturbation as demonstrated in Fig.1c). This argument consistently indicates that the topological phase ensured by the PHS is distinguished by the Z_2 invariant.

In two and three dimensions, we have multiple one-dimensional Z_2 invariants corresponding to possible C_{ij} in Fig.1b)-c). When the Z_2 invariant $(-1)^{\nu[\mathrm{C}_{ij}]}$ is non-trivial, we have a gapless state on the surface perpendicular to C_{ij}: By fixing the momenta along the surface perpendicular to C_{ij}, the part of the system can be considered as a one-dimensional gapful system [16]. Therefore, from the topological argument above, it is concluded that there exist a gapless mode on the surface.

Following the classification in [10], the two-dimensional Chern numbers ν_Ch also characterize the topological phase of the TRSB superconductors. Now make a connection between the Chern number ν_Ch and the Z_2 invariants in two dimensions. ν_Ch is defined by

$$\nu_\mathrm{Ch} = \frac{1}{2\pi}\int_{T^2} \mathcal{F}^{(-)}(\boldsymbol{k}), \tag{12}$$

where $\mathcal{F}^{(-)}(\boldsymbol{k})$ is the field strength of $A_i^{(-)}(\boldsymbol{k})$, and T^2 the two-dimensional BZ in Fig.1b) [1]. Noting that the field strength $\mathcal{F}(\boldsymbol{k})$ of $A_i(\boldsymbol{k})$ is identically zero, we find $\mathcal{F}^{(-)}(\boldsymbol{k}) = \mathcal{F}^{(-)}(-\boldsymbol{k})$ from (4). Thus the Chern number is linked to the Z_2 invariants as

$$\begin{aligned}\nu_\mathrm{Ch} &= \frac{1}{2\pi}\int_{T^2} \mathcal{F}^{(-)}(\boldsymbol{k}) = \frac{1}{\pi}\int_{T^2_+} \mathcal{F}^{(-)}(\boldsymbol{k}) \\ &= \frac{1}{\pi}\oint_{\partial T^2_+} dk_i A_i^{(-)}(\boldsymbol{k}) = \nu[\mathrm{C}_{12}] - \nu[\mathrm{C}_{34}],\end{aligned} \tag{13}$$

where T^2_+ is the upper half of T^2. Consequently, from (11), we have the following relation between ν_CH and the topology of the Fermi surface,

$$(-1)^{\nu_\mathrm{Ch}} = \prod_{\alpha, i=1,2,3,4} \mathrm{sgn}\varepsilon_\alpha(\Gamma_i) = (-1)^{p_0(S_\mathrm{F})}, \tag{14}$$

where $p_0(S_\mathrm{F})$ is the number of the connected components of the Fermi surface on T^2, and in the second equality, we have used the result in [13]. This formula provide a criterion for non-zero ν_Ch: If $p_0(S_\mathrm{F})$ is odd, then ν_Ch is non-zero. This simple criterion immediately reproduces the non-zero ν_Ch for the chiral p-wave superconductor [7] since it has a single Fermi surface. In a similar manner, it is found that the Chern numbers in three dimensions are also characterized by the topology of the Fermi surface.

Now consider TRI odd-parity superconductors. Because of the time-reversal invariance Θ with $\Theta^2 = -1$, the occupied states of the BdG Hamiltonian $H(\boldsymbol{k})$ are divided into Kramers pairs, $|u_n^s(\boldsymbol{k})\rangle$ $(s = \mathrm{I}, \mathrm{II})$,

$$|u_n^\mathrm{I}(\boldsymbol{k})\rangle = \Theta|u_n^\mathrm{II}(-\boldsymbol{k})\rangle. \tag{15}$$

Since the Kramers pair, $|u_n^{\rm I}(\Gamma_i)\rangle \equiv |u_{2n}(\Gamma_i)\rangle$ and $|u_n^{\rm II}(\Gamma_i)\rangle \equiv |u_{2n+1}(\Gamma_i)\rangle$ at Γ_i share the same eigenvalue of Π, (9) leads to $(-1)^{\nu[{\rm C}_{ij}]}=1$. In other words, $(-1)^{\nu[{\rm C}_{ij}]}$ is always trivial for TRI odd-parity superconductors. However, use of the time-reversal invariance as well makes it possible to define non-trivial Z_2 invariants.

To define non-trivial Z_2 invariants, let us introduce the gauge field $A_i^{s(-)}(\boldsymbol{k})=i\sum_{n<0}\langle u_n^s(\boldsymbol{k})|\partial_{k_i}|u_n^s(\boldsymbol{k})\rangle$, and its Wilson loop $W^s[{\rm C}_{ij}]$ for the Kramers pairs $|u_n^s(\boldsymbol{k})\rangle$,

$$W^s[{\rm C}_{ij}]=\frac{1}{2\pi}\oint_{{\rm C}_{ij}}dk_iA_i^{s(-)}(\boldsymbol{k}). \qquad (16)$$

Because $W[{\rm C}_{ij}]$ is divided into $W[{\rm C}_{ij}]=W^{\rm I}[{\rm C}_{ij}]+W^{\rm II}[{\rm C}_{ij}]$, and $W^{\rm I}[{\rm C}_{ij}]=W^{\rm II}[{\rm C}_{ij}]$ from (15), we find $W^{\rm I}[{\rm C}_{ij}]=\nu[{\rm C}_{ij}]/4$. Therefore, using $(-1)^{\nu[{\rm C}_{ij}]}=1$, we obtain the quantization of $e^{2\pi iW^{\rm I}[{\rm C}_{ij}]}$ as $e^{2\pi iW^{\rm I}[{\rm C}_{ij}]}=\pm 1$. This means that different Z_2 invariants $(-1)^{\tilde{\nu}[{\rm C}_{ij}]}$ can be introduced by $(-1)^{\tilde{\nu}[{\rm C}_{ij}]}=e^{2\pi iW^{\rm I}[{\rm C}_{ij}]}$.

For these Z_2 invariants $(-1)^{\tilde{\nu}[{\rm C}_{ij}]}$, it is shown that

$$(-1)^{\tilde{\nu}[{\rm C}_{ij}]}=\prod_{\alpha}{\rm sgn}\varepsilon_{2\alpha}(\Gamma_i){\rm sgn}\varepsilon_{2\alpha}(\Gamma_j), \qquad (17)$$

under the same weak-paring assumption as (11). Here $\varepsilon_\alpha(\Gamma_i)$ is an eigenvalue of $\mathcal{E}(\Gamma_i)$, and we have set $\varepsilon_{2\alpha}(\Gamma_i)=\varepsilon_{2\alpha+1}(\Gamma_i)$ by using the Kramers degeneracy. In terms of the topology of the Fermi surface, the right hand side of (17) is expressed by $(-1)^{i_0(S_{\rm F}\cap{\rm C}_{ij})/4}$.

For the TRI odd-parity superconductors, we also have two-dimensional Z_2 invariants $(-1)^{\nu_{\rm 2dTI}}$ which were originally used to characterize topological insulators [3], and three dimensional winding number $\nu_{\rm w}$ defined in [10]. Using the formula (17), we can connect these topological invariants to the Fermi surface topology: First, since the Z_2 invariant $(-1)^{\nu_{\rm 2dTI}}$ for topological insulators is defined as a product of $e^{2\pi iW^{\rm I}[{\rm C}_{ij}]}$ [6], it is also a product of our Z_2 invariants $(-1)^{\nu[{\rm C}_{ij}]}$. Therefore, in two dimensions, the formula (17) leads to

$$(-1)^{\nu_{\rm 2dTI}}=\prod_{\alpha,i=1,2,3,4}{\rm sgn}\varepsilon_{2\alpha}(\Gamma_i)=(-1)^{p_0(S_{\rm F})/2}. \qquad (18)$$

In a similar manner, the Z_2 invariant $(-1)^{\nu_{\rm 3dTI}}$ for three dimensional topological insulators [4, 5, 6] is represented by a product of our one-dimensional Z_2 invariants $(-1)^{\nu[{\rm C}_{ij}]}$. Moreover, the winding number $\nu_{\rm w}$ satisfies $(-1)^{\nu_{\rm w}}=(-1)^{\nu_{\rm 3dTI}}$ [13]. Accordingly, we have

$$(-1)^{\nu_{\rm w}}=\prod_{\alpha,i=1,\cdots,8}{\rm sgn}\varepsilon_{2\alpha}(\Gamma_i)=(-1)^{\chi(S_{\rm F})/4}, \qquad (19)$$

where $\chi(S_{\rm F})$ is the Euler characteristics of the Fermi surface [13]. While the formulas (18) and (19) have already been reported for TRI single-band spin-triplet superconductors and multi-band odd-parity superconductors with $P=1$ [13], here they are extended to general TRI odd-parity superconductors [20]. Furthermore, owing to the PHS, we have an additional formula (17), which was not known before.

So far, we have considered full-gapped odd-parity superconductors, but our formulas (11) and (17) are applicable to a nodal odd parity superconductor as well if the TRI path ${\rm C}_{ij}$ does not intersect a node of the superconducting gap. As was discussed above, fixing the momenta along the surface perpendicular to ${\rm C}_{ij}$, we can consider the part of the system as a one-dimensional gapful odd-parity superconductor. When the Z_2 invariant $(-1)^{\nu[{\rm C}_{ij}]}$ or $(-1)^{\tilde{\nu}[{\rm C}_{ij}]}$ is non-trivial, a gapless surface state is predicted on the surface perpendicular to ${\rm C}_{ij}$.

To conclude, in this letter, we present a description of topological odd-parity superconductors in terms of the Fermi surface topology in the normal state. All the topological invariants for odd-parity superconductors are directly related to the topology of the Fermi surface by (11) and (14) for the TRSB case, and (17), (18) and (19) for the TRI one, respectively, which provide simple criteria for topological odd-parity superconductors.

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[18] In the classification of [10], the Hamiltonian is in class D class for the TRSB case, and in class DIII for TRI one,

respectively.
[19] In this letter, the degeneracy of the Fermi surface is taken into account to count $i_0(S_{\mathrm{F}} \cap \mathrm{C}_{ij})$, $p_0(S_{\mathrm{F}})$ and $\chi(S_{\mathrm{F}})$. So, they are different from those in [13] by factor 2.
[20] The recent preprint [17] also discussed the generalization of (19) for the TRI case in a different manner.